

EXPONENTIAL DECAY OF EXPANSIVE CONSTANTS

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ABSTRACT. A map f on a compact metric space is expansive if and only if f^n is expansive. We study the exponential rate of decay of the expansive constant and find some of its relations with other quantities about the dynamics, such as dimension and entropy.

1. EXPANSIVE MAPS

Let X be a compact metric space. A homeomorphism (continuous map) f is called expansive if there is $\gamma > 0$ such that $d(f^n(x), f^n(y)) < \gamma$ for all $n \in \mathbb{Z}$ ($n \geq 0$) implies $x = y$. We call the largest γ the expansive constant of f , denoted by $\gamma(f)$, as it depends on f .

In this paper we assume that f is an expansive homeomorphism, if not specified. Results and their proofs for expansive continuous maps are very similar and will be omitted.

Our discussion on expansive constants builds on the following proposition:

Proposition 1.1. *For every $n \in \mathbb{Z}$ ($n \in \mathbb{N}$ for continuous maps), f is expansive if and only if f^n is expansive.*

Proof. From the definition, it is trivial that f is expansive if and only if f^{-1} is expansive, and for $n \in \mathbb{N}$, if f^n is expansive, then f is expansive.

Now assume that f is expansive with expansive constant $\gamma(f)$. As f is continuous, there is $\epsilon > 0$ such that $d(x, y) < \epsilon$ implies $d_f^n(x, y) < \gamma(f)$, where

$$d_f^n(x, y) = \max_{0 \leq k \leq n-1} d(f^k(x), f^k(y)).$$

So $d((f^n)^k(x), (f^n)^k(y)) < \epsilon$ for all $k \geq 0$ implies $d(f^{kn+j}(x), f^{kn+j}(y)) < \gamma(f)$ for all $k \geq 0$ and $0 \leq j \leq n-1$. f^n is expansive with expansive constant no less than ϵ . \square

From this proposition, if f is expansive, we can talk about $\gamma(f^n)$, the expansive constant of f^n . It is not difficult to make the following observations:

Lemma 1.2. *If f is expansive, then the following hold.*

- (1) *(For homeomorphisms only) $\gamma(f) = \gamma(f^{-1})$.*
- (2) *For every $n \in \mathbb{N}$, $\gamma(f^n) \leq \gamma(f)$.*
- (3) *If $d(x, y) < \tau$ implies $d_f^n(x, y) < \gamma(f)$, then $\gamma(f^n) \geq \tau$.*

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It is natural to think about how $\gamma(f^n)$ varies as n increases. For Lipschitz maps we have a very rough estimate:

Lemma 1.3. *If f is expansive and Lipschitz with Lipschitz constant $L(f)$, then $L(f) > 1$.*

Proposition 1.4. *Let f be expansive and Lipschitz with Lipschitz constant $L(f)$, then for every $n \in \mathbb{N}$, $\gamma(f^n) \geq \gamma(f) \cdot (L(f))^{-(n-1)}$.*

Proof. $d(x, y) \leq \gamma(f) \cdot (L(f))^{-(n-1)}$ implies $d_f^n(x, y) \leq \gamma(f)$. \square

Corollary 1.5. *If f is a bi-Lipschitz homeomorphism with Lipschitz constants $L(f)$ and $L(f^{-1})$, then for every $n \in \mathbb{N}$,*

$$\gamma(f^n) \geq \gamma(f) \cdot \min_{0 \leq j \leq n} \max\{(L(f))^{-j}, (L(f^{-1}))^{-(n-j)}\}$$

2. EXPONENTIAL DECAY OF EXPANSIVE CONSTANTS

As in the usual way to study a quantity about the dynamics, we consider the asymptotic exponential decay rate of the expansive constant $\gamma(f^n)$.

Definition 2.1. If f is expansive, then let

$$h_E^+(f) = \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \gamma(f^n)$$

$$h_E^-(f) = \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \gamma(f^n)$$

From now on we use h_E^* to denote either h_E^+ or h_E^- , when the argument works for both cases. Some simple facts about them are listed below.

Lemma 2.2. *If f is expansive, then*

- (1) $h_E^*(f) \geq 0$.
- (2) $h_E^-(f) \leq h_E^+(f)$.
- (3) (For homeomorphisms only) $h_E^*(f) = h_E^*(f^{-1})$.

Proposition 2.3. *If f is expansive, then for every $n \in \mathbb{N}$, $h_E^+(f^n) \leq nh_E^+(f)$ and $h_E^-(f^n) \geq nh_E^-(f)$.*

Proof.

$$h_E^+(f^n) = \limsup_{k \rightarrow \infty} -\frac{1}{k} \log \gamma(f^{kn}) \leq n \limsup_{j \rightarrow \infty} -\frac{1}{j} \log \gamma(f^j) = nh_E^+(f).$$

The other one is analogous. \square

Proposition 2.4. *If f is continuous map that is expansive and Lipschitz, then $h_E^*(f) \leq \log L(f)$. If f is homeomorphism that is expansive and bi-Lipschitz, then*

$$h_E^*(f) \leq \frac{\log L(f) \cdot \log L(f^{-1})}{\log L(f) + \log L(f^{-1})}.$$

In particular, if $L = \max\{L(f), L(f^{-1})\}$, then $h_E^(f) \leq \frac{1}{2} \log L$.*

Proof. This is a corollary of Proposition 1.4 and Corollary 1.5. \square

Proposition 2.5. *Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be expansive and there is a bi-Lipschitz conjugacy $h : X \rightarrow Y$ between them. Then $h_E^*(f) = h_E^*(g)$.*

Proof. For every n and k , $d(g^{kn}(x), g^{kn}(y)) < (L(h))^{-1} \cdot \gamma(f^n)$ implies

$$d(h(f^{kn}(h^{-1}(x))), h(f^{kn}(h^{-1}(y)))) < \gamma(f^n),$$

which provides $h^{-1}(x) = h^{-1}(y)$, hence $x = y$. So $\gamma_n(g) \leq (L(h))^{-1} \cdot \gamma(f^n)$. Similar argument shows $\gamma(f^n) \leq (L(h^{-1}))^{-1} \cdot \gamma_n(g)$. Taking limits, we have $h_E^*(f) = h_E^*(g)$. \square

Corollary 2.6. $h_E^*(f)$ is invariant under strongly equivalent metrics.

3. RELATIONS WITH THE EXPONENTIAL DECAY OF LEBESGUE NUMBERS

The exponential decay of Lebesgue numbers has been discussed in [2]. We are somewhat surprised by the relation between it and the decay of expansive constants we observe.

Recall that for every open cover \mathcal{U} of a compact metric space X , the Lebesgue number $\delta(\mathcal{U})$ is defined as the largest positive number such that every $\delta(\mathcal{U})$ ball is covered by some element of \mathcal{U} . Let f be a continuous map on X . Let $\mathcal{U}_f^n = \bigvee_{j=0}^{n-1} f^{-j}(\mathcal{U})$ and $\delta_n(f, \mathcal{U}) = \delta(\mathcal{U}_f^n)$. Define

$$h_L^-(f, \mathcal{U}) = \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \delta_n(f, \mathcal{U}),$$

$$h_L^+(f, \mathcal{U}) = \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \delta_n(f, \mathcal{U}),$$

$$h_L^-(f) = \sup_{\mathcal{U}} h_L^-(f, \mathcal{U})$$

and

$$h_L^+(f) = \sup_{\mathcal{U}} h_L^+(f, \mathcal{U}).$$

Theorem 3.1. *If f is expansive, then $h_E^*(f) \leq h_L^*(f)$.*

Proof. Let \mathcal{U} be an open cover such that $\text{diam} \mathcal{U} < \gamma(f)$. Then for each $n > 0$, $d(x, y) < \delta_n(f, \mathcal{U})$ implies $d(f^j(x), f^j(y)) < \text{diam} \mathcal{U} < \gamma(f)$ for $0 \leq j \leq n-1$. So if $d(f^{kn}(x), f^{kn}(y)) < \delta_n(f, \mathcal{U})$ for every k , then $d(f^m(x), f^m(y)) < \gamma(f)$ for every $m = kn + j$, which runs over all integers. This forces $x = y$ as $\gamma(f)$ is the expansive constant of f . So $\gamma(f^n) \leq \delta_n(f, \mathcal{U})$. Take the limit and we obtain $h_E^*(f) \leq h_L^*(f, \mathcal{U}) \leq h_L^*(f)$. (The last relation is in fact an equality from [2, Corollary 3.9]) \square

4. RELATIONS WITH ENTROPY AND DIMENSION

The most important fact we observe is that the product of $h_E^*(f)$ and box dimension also bounds topological entropy. By Theorem 3.1, this bound is (strictly, see Theorem 4.3 for example) better than [2, Theorem 4.7]. But this result only makes sense when f is expansive.

Theorem 4.1. *Let f be expansive on a compact metric space X . $\dim_B^\pm X$ are the upper and lower box dimensions of X and $h(f)$ is the topological entropy of f . Then $h_E^-(f) \cdot \dim_B^+ X \geq h(f)$ and $h_E^+(f) \cdot \dim_B^- X \geq h(f)$.*

Proof. We only show the first inequality. Proof of the other is similar.

The result is trivial if $h(f) = 0$. Assume $h(f) > 0$. Take any $\lambda > \dim_B^+ X$. There is $\varepsilon_0 > 0$ such that $\varepsilon < \varepsilon_0$ implies that there is an open cover \mathcal{U} of X such that $\text{diam} \mathcal{U} < \varepsilon$ and $|\mathcal{U}| \leq \varepsilon^{-\lambda}$. If n large enough such that $\exp nh(f) = \exp h(f^n) >$

$\varepsilon^{-\lambda}$, then \mathcal{U} is not a generator under f^n (see for example, [1, Section 5.6]), as there are at most $\varepsilon^{-k\lambda}$ elements in $\bigvee_{j=0}^{k-1} f^{-jn}(\mathcal{U})$, which makes

$$h(f^n, \mathcal{U}) = \lim_{k \rightarrow \infty} \frac{1}{k} H\left(\bigvee_{j=0}^{k-1} f^{-jn}(\mathcal{U})\right) \leq \log(\varepsilon^{-\lambda}) < h(f^n).$$

There is $A \in \bigvee_{j=-\infty}^{\infty} f^{-jn}(\mathcal{U})$ that contains at least two points, say, x and y . Then for every $j \in \mathbb{Z}$, $d(f^{jn}(x), f^{jn}(y)) < \varepsilon$. ε is not an expansive constant for f^n and $\gamma(f^n) < \varepsilon$.

Now take $N > -\frac{\lambda \log \varepsilon_0}{h(f)}$. For every $n > N$, take $\varepsilon < \varepsilon_0$ such that $\exp(n-1)h(f) < \varepsilon^{-\lambda} < \exp nh(f)$. Then

$$-\frac{\log \gamma(f^n)}{n} > -\frac{\log \varepsilon}{n} > \frac{(n-1)h(f)}{n\lambda}$$

Take the lower limit we get $h_E^-(f) \cdot \lambda > h(f)$. The result follows since λ is arbitrarily chosen. \square

The idea of the proof is a byproduct of the following problem considered by the author.

Problem. *Let f be an expansive homeomorphism on a compact metric space. What is the smallest possible number of elements in a generator? How is this number related to other properties of f ?*

It is sure that this number should be at least $\exp h(f)$. So the best result we can expect is the integer no less than $\exp h(f)$, and this is the case for full shifts. However, even for subshifts of finite types we have no idea at this moment.

As for subshifts of finite types, we observe the following fact.

Proposition 4.2. *Let σ_A be a subshift of finite type. Let $q > 1$ and the metric on Ω_A be defined by $d(\omega, \omega') = q^{-\min\{|j| : \omega_j \neq \omega'_j\}}$. Then $h_E^*(\sigma_A) \cdot \dim_H \Omega_A = h(\sigma_A)$, where $\dim_H \Omega_A$ is the Hausdorff dimension of Ω_A .*

Remark. It is well-known that under the above assumptions,

$$\dim_B^+ \Omega_A = \dim_B^- \Omega_A = \dim_H \Omega_A = \frac{2}{\log q} \lim_{k \rightarrow \infty} \frac{1}{k} \log \|A^k\|.$$

(or $\frac{1}{\log q} \lim_{k \rightarrow \infty} \frac{1}{k} \log \|A^k\|$ for one-sided shifts.)

Proof. We only show the result for two-sided shifts. Proof for one-sided shifts is analogous.

We know that $h(\sigma_A) = \lim_{k \rightarrow \infty} \frac{1}{k} \log \|A^k\|$. If $h(\sigma_A) = 0$, the result is trivial. Otherwise, it is enough to show that $h_E^*(\sigma_A) = \frac{\log q}{2}$.

If $d(\omega, \omega') < q^{-n}$, then we must have $\omega_j = \omega'_j$ for all $-n \leq j \leq n$. So $d(\sigma_A^{k(2n-1)}(\omega), \sigma_A^{k(2n-1)}(\omega')) < q^{-n}$ for all $k \in \mathbb{N}$ implies $\omega = \omega'$, hence $\gamma(\sigma_A^{2n-1}) \geq q^{-n}$. In fact, $\gamma(\sigma_A^l) \geq q^{-n}$ for all $l \leq 2n-1$. So $h_E^-(\sigma_A) \geq \frac{\log q}{2}$.

By Theorem 4.1, $h_E^+(\sigma_A) \leq \frac{\log q}{2}$. The result follows. \square

The above result is not so difficult but very interesting. It seems that for certain expansive dynamical systems the topological entropy may also be given by the product $h_E^*(f) \cdot \dim_H(X)$. Though the box dimensions may be different from the Hausdorff dimension when the metric is changed, we believe that the result involving Hausdorff dimension is always true. Considering Corollary 2.6, the result is true for

many other commonly used metrics on symbolic spaces. A proof for all equivalent metrics is in process.

Moreover, as every Anosov diffeomorphism is expansive and has Markov partition, we may expect the following result:

Conjecture 1. *Let f be an Anosov diffeomorphism on a m -dimensional compact manifold, then $h(f) = mh_E^*(f)$.*

It would be a big surprise if the conjecture is true. Nevertheless, the following fact might boost our confidence, at least a little bit.

Theorem 4.3. *The conjecture is true for hyperbolic linear automorphisms on the 2-torus with the standard metric.*

Proof. Let $\lambda > 1$ and λ^{-1} be the eigenvalues. For every $\mu > \lambda$, there is a metric that is strongly equivalent to the standard metric, such that the diffeomorphism f is a bi-Lipschitz map with Lipschitz constants $L(f) < \mu$ and $L(f^{-1}) < \mu$. By Proposition 2.4 and Corollary 2.6, $h_E^*(f) < \frac{1}{2} \log \mu$. Since μ is arbitrarily taken and by Theorem 4.1, we have $h_E^*(f) = \frac{1}{2} \log \lambda = \frac{1}{2} h(f)$. \square

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